Multiple effects of gradient coupling on network synchronization

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(Received 14 January 2008; revised manuscript received 9 April 2008; published 16 May 2008)

Recent studies have shown that the synchronizability of complex networks can be significantly improved by gradient or asymmetric couplings, and increase of the gradient strength could enhance the network synchronizability monotonically. Here we argue and demonstrate that, for a typical complex network, there could be an optimal gradient where the maximum network synchronizability is achieved. That is, large gradient may deteriorate synchronization. We attribute the suppressing effect of gradient coupling to the phenomenon of network breaking and show that, comparing with sparse homogeneous networks, dense heterogeneous networks suffer less from network breaking and, consequently, benefit more from large gradient in improving synchronization. The findings are supported by indirect simulations of eigenvalue analysis and direct simulations of coupled nonidentical oscillators.

DOI: 10.1103/PhysRevE.77.056208 PACS number(s): 05.45.Xt, 89.75.−k

Complex networks have attracted a great deal of interest since the discoveries of the small-world [1] and scale-free [2] properties. Roughly, small-world networks are characterized by a locally highly regular connecting structure and a globally small network distance, while the defining characteristic of scale-free networks is a power-law distribution $P(k) \sim k^{-\gamma}$ in the node degree. Signatures of small-world and scale-free networks have been discovered in many natural and man-made systems [3–5], and they constitute the cornerstones of modern network science.

At a systems level, synchronization is one of the most common dynamical processes. For instance, in biology, synchronization of oscillator networks is fundamental [6]. In a computer network designed for large scale, parallel computation, to achieve synchronous timing is essential. Recent studies of the synchronizability of complex networks have revealed that small-world and scale-free networks, due to their small network distances, are generally more synchronized than regular networks [7,8] and, with weighted and asymmetric couplings, the synchronizability (a measure of synchronization propensity to be explained later in this paper) of scale-free networks can be significantly improved and, in general, can be much higher than that of small-world networks [10–16].

Links in practical networks are usually granted with weight and orientation [4], which, in terms of network synchronization, refers to coupling strength and coupling direction, respectively [6]. If the pair of directed couplings between two connected nodes has different weights, they are called asymmetric couplings. Regarding each pair of asymmetric couplings as a superposition of an undirected symmetric coupling and a directed gradient coupling, we thus are able to explore the functions of asymmetric couplings in network synchronization by focusing on only the gradient part [16]. Previous studies have shown that, to improve network synchronization, both the direction and weight of the gradient should be properly arranged, based on network information such as node degree [10,11] or betweenness [12]. In Refs. [10,11] it is shown that, by setting gradient flow from the larger-degree node to the smaller-degree node on each link, the synchronizability of scale-free networks can be significantly improved and outperforms that of homogeneous networks. Setting the gradient direction like this, in Refs. [11,16] it is shown that the network synchronizability can be monotonically improved by increasing the weight of the gradient. The enhancing effect of gradient coupling is further highlighted in Ref. [13], where nodes are suggested to be connected by only gradient couplings, i.e., the one-way-coupling configuration.

While most of the previous studies are focusing on the enhancing role of gradient coupling on network synchronization, there are accumulating evidences showing the opposite: Large gradient deteriorates synchronization sometimes. For example, in Refs. [10,15] it is observed that as the gradient increases, network synchronizability is first enhanced, gradually reaching to its maximum at an optimal gradient; then, after this optimal value, increase of gradient will suppress synchronization. The similar phenomenon is also briefly reported in Ref. [11]; there it is found that the suppressing effect of large gradient has a close relationship to the network topology. Despite these observations, a detailed study on the effects of gradient coupling on network synchronizability is still absent.

Our main findings are the following: (1) The suppressing effect of large gradient is induced by the phenomenon of network breaking; (2) small, dense, heterogeneous networks have a smaller breaking probability than large, sparse, homogeneous networks; (3) while large gradient suppressing the propensity for global synchronization, the propensity for partial synchronization is enhanced.

We consider oscillator networks of the following form:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j=1}^{N} G_{i,j}\mathbf{H}(\mathbf{x}_j), \quad i = 1, \ldots, N, \tag{1}$$

where $\mathbf{F}(\mathbf{x}_i)$ governs the local dynamics of uncoupled node i, $\mathbf{H}(\mathbf{x})$ is the coupling function, ε is the coupling strength, and

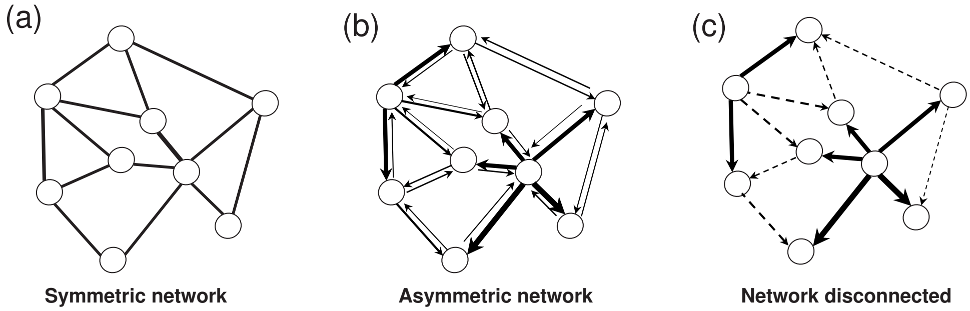

FIG. 1. A schematic plot on the transformation from (a) an unweighted symmetric network [generated by replacing G with A in Eq. (2)] to (b) a weighted asymmetric network [generated by a finite value of β in Eq. (2)] and to (c) a directed tree-structure network of degenerative type [generated by setting $\beta\to\infty$ in Eq. (2)]. The dashed lines in (c) denote the links that are gradually diminished due to the increased gradient strength.

$G_{i,j}$ is an element of the coupling matrix $\mathbf{G}$ which takes the form [14–16]

$$G_{i,j} = -\frac{A_{i,j}k_j^{\beta}}{\sum_{j=1}^{N} A_{i,j}k_j^{\beta}} \quad \text{for } i \neq j. \tag{2}$$

with k_i the degree of node i and $A=\{a_{i,j}\}$ the adjacency matrix of the network, $a_{i,j}=1$ if nodes i and j are connected, $a_{i,j}=0$ otherwise, and $a_{i,i}=0$. To keep the synchronization state a solution of the system, we choose $G_{i,i}=1$.

It is worth noting that the parameter β in Eq. (2) modulates both the direction and the weight of the gradient coupling [16]. Generally speaking, if $\beta>0$, the gradient will flow from larger-degree to smaller-degree node, and vice versa. By tuning the value $|\beta|$, we are also able to adjust the gradient weight: Larger $|\beta|$ generates larger coupling gradient [16]. We note that in changing β the total coupling cost of the network is keeping constant, only the weight of the couplings is redistributed. We also note that many coupling schemes proposed in the literature can be generated from Eq. (2) by adopting different values of β. For instance, the symmetric network model in Refs. [7,9] can be realized by replacing G with A; the asymmetric network model in Ref. [10] can be reproduced by setting $\beta=0$; and the tree-structure network model proposed in Ref. [13] in principle can be achieved by setting $\beta\to\infty$ [17]. A schematic plot on the transformation of Eq. (2) into different network models is present in Fig. 1.

The limiting case of $\beta\to\infty$ in Eq. (2) is of special interest [Fig. 1(c)], as it stands for the extreme case of gradient coupling: The one-way-coupling structure. In this structure, each node receives coupling only from its largest-degree neighbor. (Strictly speaking, each node is receiving coupling from the neighbor who has the largest node scalar $h_i=k_i^{\beta}\Sigma_{l\in V_i}k_l^{\beta}$, where V_i denotes the set of neighboring nodes of i. However, for complex networks of null degree correlation, statistically we have $h_i\sim k_i$. See Ref. [16] for details.) As such, the network consists of N nodes and $N-1$ directed links [17]. If the network is nondegenerate (connective), all of the nodes will be organized into a unique tree structure, with the largest-degree node of the network located at the root. For this type of network, increased β will enhance synchronization monotonically, with the maximum synchronizability [13,16] achieved at $\beta\to\infty$. Otherwise, if the network is degenerative, i.e., the largest-degree nodes are not directly connected, the network will be broken when $\beta\to\infty$ [Fig. 1(c)]. In this degenerative case, each of the largest-degree nodes will generate an isolating "subtree," and the network is broken into several disconnected subnetworks, i.e., forming a forest. Once forest is generated, the network can never be globally synchronized whatever the coupling strength, manifesting the destructive role of the gradient.

It is important to note that, when forest emerges, although global synchronization is unreachable, the nodes are self-organized into several synchronous clusters according to the subnetworks. Nodes on the same subnetwork are synchronized, but are not if they are on different subnetworks. This cluster phenomenon, however, is not restricted to the special case of "forest" network, i.e., the limiting case of $\beta\to\infty$, but can also be found in general degenerative networks which have finite β (as will be shown later by simulations). In this regard, we say that the degenerative network has the potential to be developed into clusters, but this potential can only be induced by proper gradient, say, for example, when the gradient parameter is larger than some threshold value β_o.

To have a global picture on the multiple effects of the gradient coupling, we increase β from 0 gradually and check the variation of the network synchronizability. Network synchronizability can be evaluated by the method of master stability function (MSF) [8], which states that a network is generally more synchronized when the spread of the eigenvalue spectrum of its coupling matrix is narrow. In particular, let $0=\lambda_1\leq\lambda_2\cdots\leq\lambda_N$ be the eigenvalue spectrum of the coupling matrix G. Then the smaller the ratio $R\equiv\lambda_N/\lambda_2$, the more likely synchronous dynamics is to occur on the network. In general, asymmetric coupling matrix G possesses complex eigenvalues. However, for our coupling scheme of Eq. (2) it can be proven that the eigenvalues of G are real and non-negative [16].

In simulation, we generate a scale-free network of $N=3\times10^3$ nodes and average degree $\langle k\rangle=8$ by the generalized model introduced in Ref. [18]. In this model, the degree exponent γ can be adjusted via a parameter θ. Defining the new preferential attachment function as $p\sim(k_i+\theta)/\Sigma_j(k_j+\theta)$, it can be proven that the generated network has degree exponent $\gamma=3+\theta/m$, with $m=4$ the number of new links associ-

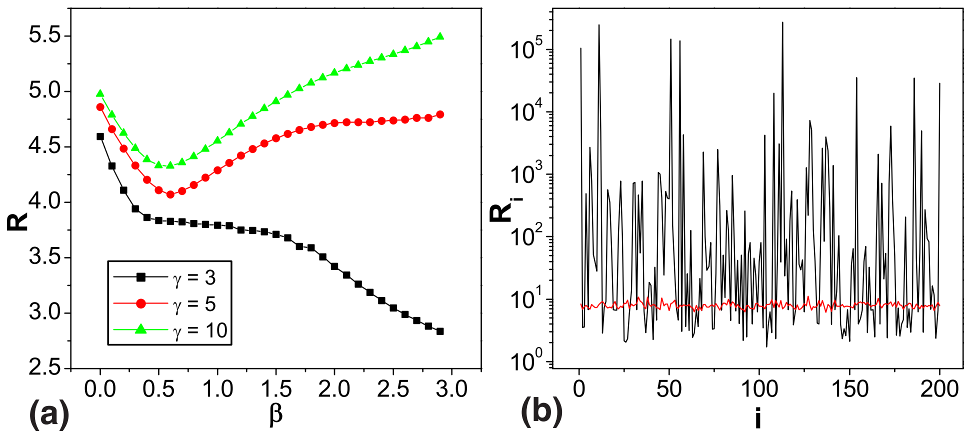

FIG. 2. (Color online) For scale-free networks of $N=3\times10^3$ nodes and average degree $\langle k\rangle=8$. (a) The variation of the eigenratio R as a function of the gradient parameter β for scale-free networks of degree exponent $\gamma=3$ (the lower curve), $\gamma=5$ (the middle curve), and $\gamma=10$ (the upper curve), respectively. The optimal gradient β_o appears at about 0.6 for the last two cases. Each data is an average result over 100 network realizations. (b) $\gamma=10$, the distribution of the individual network eigenratio R_i under parameters $\beta_1=0.2$ (the red oscillation) and $\beta_2=10$ (the black oscillation). The extreme values of R_i indicate the broken network topology in the corresponding network realizations.

ated to each new node. Using $\theta=0$, 8, and 28, we generate scale-free networks of degree exponents $\gamma=3$, 5, and 10, respectively. The variations of R as a function of β for these three networks are plotted in Fig. 2(a). It is found that, when $\gamma=3$, increasing β will enhance network synchronizability monotonically; however, when $\gamma=5$ or $\gamma=10$, as β increases from zero, the synchronizability is first enhanced, then, after reaching its maximum at about $\beta_o\approx0.6$ [20], increasing β begins to decrease synchronizability. (It should be noted that the critical degree exponent γ_c from where the crossover of synchronizability emerges is closely related to the network parameters. In general, we have smaller γ_c in large, sparse networks. For the parameter setting of Fig. 1, numerically we find that $\gamma_c\approx5$.) Another interesting finding in Fig. 2(a) is, for any fixed value of β, that network of small γ always assumes a higher synchronizability than a network of large γ, no matter whether they are degenerative or nondegenerative.

We go on to investigate the mechanism governing the crossover of R in networks of large γ. Our approach is to analyze the network performance individually at each side of the optimal gradient β_o. Using $\gamma=10$, in Fig. 2(b) we plot the distributions of R_i for networks of gradient parameters $\beta_1=0.2<\beta_o$ and $\beta_2=10>\beta_o$, respectively. It is found that, for $\beta_1=0.2$, R_i is oscillating around its mean value $R\approx4.6$ with very small fluctuations; whereas for $\beta_2=10$, R_i oscillate widely and occasionally bursts into some extreme values of order 10^3. As β increases, these bursts will diverge to infinity gradually. Since $R\equiv\lambda_N/\lambda_2$ and λ_N is bounded [11,16], the divergence in R_i thus gives $\lambda_2\to0$, an indicator for network breaking. Now the effects of gradient coupling on network synchronizability are clear: Increasing gradient indeed can improve synchronization, but only for nondegenerative networks; while for degenerative networks, increasing gradient will also enhance the breaking effect which suppresses synchronization. The optimal gradient shown in homogeneous networks is just a balance between the enhancing and suppressing effects.

By knowing that synchronization suppression at large gradient is induced by network breaking, we next investigate the relationship between the breaking probability and the network parameters. To facilitate the analysis, we consider again the limiting case $\beta\to\infty$ in Eq. (2). As shown in Fig. 1(c), network breaking occurs when there is more than one local-maximum-degree node that exists on the network. To break a network, the "breaking nodes" do not have to be of very large degree, they are only required to have the largest degree when compared with their nearest and next-nearest (NN) neighbors. Once broken, the network will be divided into several tree-structure subnetworks, with each subnetwork led by one of such "breaking nodes." By simulation, we plot in Figs. 3(a) and 3(b) the variation of P_b as functions of γ and $\langle k\rangle$, respectively. It is clearly shown that, in comparing sparsely connected homogeneous networks, densely

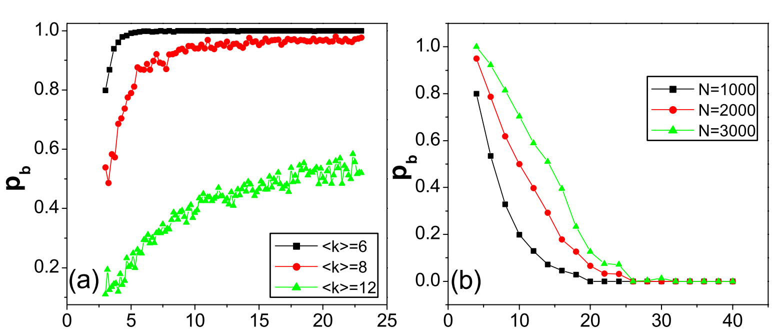

FIG. 3. (Color online) For scale-free network of $N=3\times10^3$ nodes and under the limiting case of $\beta\to\infty$ in Eq. (2). (a) The probability of network breaking P_b as a function of the degree exponent γ. (b) $\gamma=7$, P_b versus the average degree $\langle k\rangle$. Each data is an averaged result of 1000 network realizations.

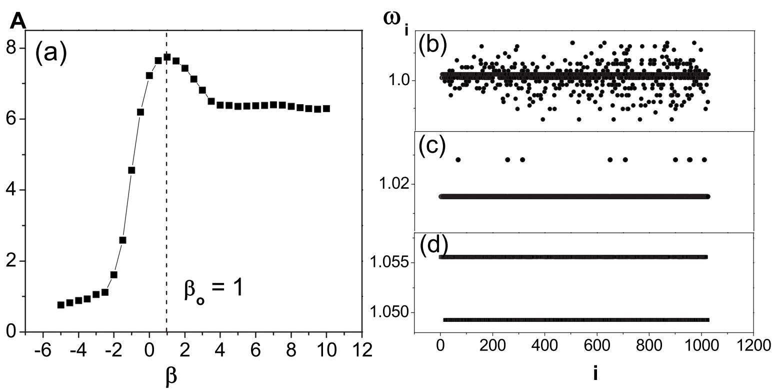

FIG. 4. For scale-free networks of $N=2\times10^3$ nonidentical Rössler oscillators and average degree $\langle k\rangle=6$. (a) For $\varepsilon=0.15$, the variation of the mean-field amplitude A as a function of the gradient parameter β. Network synchronization is optimized at about $\beta_o\approx1$. Each data is an averaged result of 50 network realizations. For a selected network realization from (a), the frequency distribution of the oscillators under gradient parameters $\beta=0.1$ (b), $\beta=1$ (c), and $\beta=10$ (d).

connected heterogeneous networks are more difficult to break [19]. Figure 3(b) also indicates an increased breaking probability for larger networks. These findings are coincident with the results in Fig. 2; both tell that heterogeneous networks benefit more from large gradient in improving synchronization.

The trend that P_b is increasing with γ and decreasing with $\langle k\rangle$ can be qualitatively analyzed, as follows. For a randomly selected node of degree k, the number of its NN neighbors is $n_2=k\langle k'\rangle-k$, with $\langle k'\rangle=\int kP'(k)dk$ and $P'(k)\sim k^{1-\gamma}$ the average degree and the degree distribution of the nearest neighbors, respectively. The probability for a randomly selected NN neighbor to have degree larger than k is $P_b^2(k)=\int_k^{k_{\max}}P''(k)dk$. Here $k_{\max}$ denotes the largest degree of the network and $P''(k)\sim k^{2-\gamma}$ is the degree distribution of the NN neighbors. Meanwhile, the probability for a nearest neighbor to have degree larger than k is $P_b^1(k)=\int_k^{k_{\max}}P'(k)dk$. Therefore, the chance for the selected node to be the largest degree node among both its nearest and NN neighbors is $P_b(k)=1-n_2P_b^2-kP_b^1$, which, summarizing over k, gives the breaking probability of the network

$$P_b=\int P_b(k)P(k)dk-1=N[1-\alpha_1\alpha_2m^2+(\alpha_1-\alpha_3)m]-1, \tag{3}$$

where $\alpha_1=(\gamma-1)/(2\gamma-5)$, $\alpha_2=(\gamma-2)/(2\gamma-3)$, and $\alpha_3=(\gamma-1)/(2\gamma-4)$ are parameters dependent on γ. The unity subtracted in Eq. (3) accounts for the unique largest-degree node of the network. From Eq. (3) it is not difficult to prove that $dP_b/d\gamma>0$ and $dP_b/dm<0$, indicating a monotonic increase of the breaking probability for larger γ or smaller $\langle k\rangle$ [since $\langle k\rangle\approx m(\gamma-1)/(\gamma-2)$]. A detailed analysis on the breaking probability will be given elsewhere [21].

The results exemplified in Figs. 2 and 3 are based on eigenvalue and topology analysis. It is useful to examine the gradient effects in real oscillator networks. For this purpose we have checked the synchronization of scale-free networks of coupled nonidentical chaotic Rössler oscillators, a typical model employed in studying network synchronization [8,10–12]. The dynamics of a single oscillator is described by $\mathbf{F}_i(\mathbf{x}_i)=[-\omega_iy_i-z_i,\omega_ix_i+0.15y_i,z_i(x_i-8.5)+0.4]$, where ω_i is the natural frequency of the ith oscillator. In simulations we choose ω_i randomly from range [0.9,1.1], so as to make the oscillators nonidentical. The coupling function is chosen to be $\mathbf{H}(\mathbf{x})=\mathbf{x}$. The degree of synchronization can be characterized by monitoring the amplitude A of the mean field $X(t)=\sum_{i=1}^{N}x_i(t)/N$ [10]. For small coupling strength ε, $X(t)$ oscillates irregularly and A is approximately zero, indicating lack or a lower degree of synchronization. As the coupling parameter is increased, synchronization sets in and A is increased gradually from zero (nonsynchronous state) to its maximum (synchronous state). By $\varepsilon=0.15$, we plot in Fig. 4(a) the variation of A as a function of β. It is found that as β increases A is first increased, and reaching its maximum at about $\beta_o\approx1$, manifesting the constructive role gradient coupling. Then, as β increases from β_o, A begins to decrease, manifesting the destructive role of gradient coupling. An interesting finding is that increasing β further does not decrease A continuously, the system will always keep on high coherence which is about $A\approx8$. Please note that for the adopted coupling function $\mathbf{H}$, the MSF function [8] has an open synchronizable region, which confirms that the decrease of A at $\beta\geq\beta_o$ in Fig. 4(b) is induced exclusively by network breaking (instead of the shortest wave instability [8]).

The finding that A keeps on large values at very large β indicates that, despite the increased probability of network breaking, nodes are still strongly correlated, but with a different manner. To gain insight, we randomly select a network realization in Fig. 4(a), and plot the frequency distribution ω_i of the oscillators under different gradient parameters: $\beta=0.1$ in Fig. 4(b), $\beta=1$ in Fig. 4(c), and $\beta=10$ in Fig. 4(d). It is found that, for small gradient [Fig. 4(b)], ω_i is distributed randomly around the mean value $\omega=1$, indicating a low system coherence; around the optimal gradient [Fig. 4(c)], most of the oscillators are synchronized to the same frequency, indicating a higher system coherence; interestingly, for very large gradient [Fig. 4(d)], the oscillators are divided into two clusters of similar size, frequency in one cluster is different to that in the other one. Figure 4(d) indicates that even in a breakable network, the system coherence can be still very high, due to the existence of synchronized clusters.

In summary, we have studied the multiple effects of coupling gradient on network synchronization, and investigated the dependence of these effects to the network parameters. By this study, we wish to bring note to the utilization of gradient coupling for network synchronization, especially for coupling schemes established on local network information [10,11,14–16].

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